SOLUTION OF EQUATION OF ADMIXTURE TRANSFER IN A CYLINDRICAL CHANNEL WITH THE DEPENDENCE OF DIFFUSION COEFFICIENT ON CONCENTRATION AND GAS EXCHANGE WITH WALL TAKEN INTO ACCOUNT

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A more accurate linearization method is presented of the two-dimensional equation for convective-diffusive transfer. The solution of the transfer equation is obtained for a variable diffusion coefficient which depends on concentration.

The equation of convective diffusion is analyzed to solve a number of problems either in hydrodynamics or chemical kinetics and to study chemotropic converters or mass transfer during osmosis, as well as other cases.

In the case of a stationary flow of incompressible fluid in a cylindrical half-bounded channel the equation for convective-diffusive mass transfer can be written as

$$\frac{\partial}{\partial x}\left[D(c)\frac{\partial c}{\partial x}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[rD(c)\frac{\partial c}{\partial r}\right]=u_0\frac{\partial c}{\partial x}+\frac{\partial c}{\partial t}. \tag{1}$$

The solution of the equation is required under the following initial and boundary conditions:

$$c(x,\ r,\ 0)=c_1;$$
$$c(0,\ r,\ t)=f(t),\quad t>0,\quad 0\leqslant t\leqslant\infty; \tag{2}$$
$$\left(\frac{\partial c}{\partial r}+\beta c\right)_{r=r_0}=\beta c_0,$$

where $\beta = A/\varphi$.

Equation (1) is nonlinear. Such equations are, as a rule, linearized by replacing the function D(c) by some average value of it, D_{av}. Carrying out such linearization one obtains

$$D_{av}\left[\frac{\partial^2 c}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{\partial c}{\partial r}\right)\right]=u_0\frac{\partial c}{\partial x}+\frac{\partial c}{\partial t}. \tag{3}$$

The assumption that the diffusion coefficient is constant with changing concentration is only valid if the concentration varies slightly. Otherwise, errors which are difficult to estimate arise. By differentiating (1), one obtains

$$D_x^1(c)\frac{\partial c}{\partial x}+D(c)\frac{\partial^2 c}{\partial x^2}+D(c)\frac{\partial c}{\partial r}+D_r^1(c)\frac{\partial c}{\partial r}+D(c)\frac{\partial^2 c}{\partial r^2}=u_0\frac{\partial c}{\partial x}+\frac{\partial c}{\partial t}. \tag{4}$$

Thus, if one assumes that $D(c) = D_{av}$, one sets by the same token the first and the fourth terms of the above equation equal to 0.

An improved linearization of Eq. (1) is considered below, as well as the solving of the boundary-value problem.

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The integral replacement is carried out:

$$v = \int_{c_0}^{c} D(c)\,dc = \psi(c). \tag{5}$$

Then (1) becomes

$$D(c)\left[\frac{\partial^2 v}{\partial x^2} + \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial v}{\partial r}\right)\right] + u_0\,\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}. \tag{6}$$

Equation (6) is now linearized by setting $D(c) = D_{av}$:

$$D_{av}\left[\frac{\partial^2 v}{\partial x^2} + \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial v}{\partial r}\right)\right] = u_0\,\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}. \tag{7}$$

The initial and boundary conditions now become as follows:

$$v(x,\ r,\ 0) = \psi(c_1) = v_1;\quad v(0,\ r,\ t) = \psi[f(t) = G(t);\quad 0 \leqslant t < +\infty;$$
$$\left(\frac{\partial v}{\partial r} + \beta D(c)\right)\Big|_{r=r_0} = \beta c_0 D(c). \tag{8}$$

The proposed linearization takes into account to some extent the first and the third terms of Eq. (4).

The last boundary condition (8) is also nonlinear. We write it as

$$\frac{\partial v}{\partial r} + \beta D(c)(c - c_0) = 0. \tag{9}$$

One now sets

$$D(c)(c - c_0) = kv. \tag{10}$$

Then by using (5) one has

$$\frac{\partial v}{\partial r} + \beta k \int_{c_0}^{c} D(c)\,dc = 0. \tag{11}$$

To linearize the condition (9) we proceed as follows. It is assumed, following [2, 4], that the diffusion coefficient is a power of the concentration and is expressed by the formula

$$D = D_0(c - c_0)^{k-1}. \tag{12}$$

Then by inserting (11) into (5), one obtains

$$v = \int_{c_0}^{c} D(c)\,dc = D_0 \int_{c_0}^{c} (c - c_0)^{k-1}\,dc = \frac{D_0}{k}\,(c - c_0)^k. \tag{13}$$

By using (13), the condition (11) can be written as follows:

$$\frac{\partial v}{\partial r} + \beta c(c - c_0)^k = 0 \tag{14}$$

or

$$\frac{\partial v}{\partial r} + \alpha_1 v = 0, \tag{15}$$

where $\alpha_1 = \beta k$.

The nonlinear condition (8) has thus been replaced by the linear one (15).

To solve Eq. (7) under the conditions (8) and (15) one first introduces another function:

$$v^* = v - v_1. \tag{16}$$

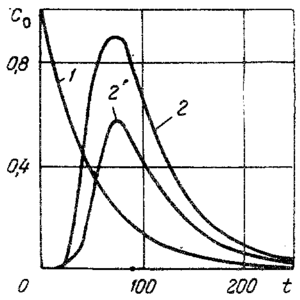

Fig. 1. Concentration curves for the constant (2') or variable (1, 2) diffusion coefficients. t, sec; c_0, mole fractions.

Then

$$D_{av}\left[\frac{\partial^2 v^*}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{\partial v^*}{\partial r}\right)\right]=u_0\frac{\partial v^*}{\partial x}+\frac{\partial v^*}{\partial t}; \tag{17}$$

$$\left.\begin{aligned} &v^*(x,\ r,\ 0)=0;\\ &v^*(0,\ r,\ t)=G(t)-v_1=G^*(t),\quad 0\leqslant t\leqslant+\infty;\\ &\left(\frac{\partial v^*}{\partial r}+\alpha_1 v^*\right)\Bigg|_{r=r_0}=-\alpha_1 v_1. \end{aligned}\right\} \tag{18}$$

Introducing the replacements

$$x=r\xi,\quad r=r_0\rho,\quad t=\frac{r_0^2\tau}{D_{av}},\quad u_0=\frac{wD_{av}}{r_0},$$

one brings the system (17)-(18) to the form

$$\frac{\partial^2 v^*}{\partial \xi^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v^*}{\partial\rho}\right)=w\frac{\partial v^*}{\partial\xi}+\frac{\partial v^*}{\partial r}; \tag{19}$$

$$v^*(\xi,\ \rho,\ 0)=0;$$

$$v^*(0,\ \rho,\ \tau)=\varphi^*(\tau)=G^*\left(\frac{D_{av}t}{r_0^2}\right),\quad 0\leqslant t<+\infty;$$

$$\frac{\partial v^*}{\partial\rho}\bigg|_{\rho=1}+Av^*=-Av_1,\quad A=\alpha r_0. \tag{20}$$

The solution of Eq. (19) together with the conditions (20) was obtained by us in [1] as

$$v=\sum_{n=1}^{\infty}\left\{\int_0^{\tau}\left[\left(\frac{2[G(\tau-\sigma)-v_1]J_1(\lambda_n)}{\lambda_n[J_1^2(\lambda_n)+J_0^2(\lambda_n)]}+J_0(\lambda_n)r_0\beta k v_1\int_0^{\tau-\sigma}\exp(-\lambda_n^2\theta)\,d\theta\right.\frac{x}{2\sigma\sqrt{\pi\sigma}r_0}\right.$$

$$\left.\times\exp\left(-\frac{u_0^2r_0^2}{4D_{av}^2}+\lambda_n\sigma-\frac{x^2}{4\sigma r_0^2}+u_0\frac{x}{2D_{av}}\right)\right]d\sigma-J_0(\lambda_n)r_0\beta k v_1\int_0^{\tau}\exp[\lambda_n^2\sigma]\,d\sigma\Bigg\}J_0(\lambda_n\rho)+v_1, \tag{21}$$

where $\tau = tD_{av}/r_0^2$.

By using the relation (13) between v and D, one finds the concentration distribution of the admixture. Substituting (21) into (13), one finds

$$c(x,\ r,\ t)=\left\{\sum_{n=1}^{\infty}\left\{\int_0^{\tau}\left[\left(\frac{2[[f(\tau-\sigma)-c_0]^k-(c_1-c_0)^k]J_0(\lambda_n)}{\lambda_n[J_1^2(\lambda_n)+J_0^2(\lambda_n)]}\right.\right.\right.\right.$$

$$\left.+J_0(\lambda_n)r_0\beta D_0c_1^k\int_0^{\tau-\sigma}\exp(-\lambda_n^2\theta)\,d\theta\right)\frac{x}{2\sigma\sqrt{\pi\sigma}r_0}\exp\left(-\frac{u_0^2r_0^2}{4D_{av}^2}+\lambda_n^2\sigma-\frac{x^2}{4\sigma r_0^2}+\frac{u_0x}{2D_{av}}\right)\Bigg]d\sigma$$

$$\left.\left.-J_0(\lambda_n)r_0(\beta)D_0c_1^k\int\exp(\lambda_n^2\sigma)\,d\sigma\right\}J_0\left(\lambda_n\frac{r}{r_0}\right)+(c_1+c_0)^k\right\}^{\overline{k}}+c_0. \tag{22}$$

The average over the radius concentration of the admixture is calculated for the equations $c_0 = c_1 = 0$; $f(t) = \exp[-0.02t]$; $D_0 = 0.01\ m^2/sec$; $k = 2$; $n = 1$; $D_{av} = 0.005\ m^2/sec$; $x = 0.3$ m; $u_0 = 0.05\ m^2/sec$; $\beta = 0.0007$.

The results of the calculations are shown in Fig. 1. The curves 1 and 2 for x = 0 or 3 m are found by using a formula arrived at by averaging the expression (22) over the radius, and the curve 2' is found for the same conditions but without linearization. It can be concluded from these curves that the proposed linearization is within the ratio 1.2-1.5 as compared with usual linearization.

NOTATION

c	is the admixture concentration, in fractions;
c_1	is the initial admixture concentration in the channel;
r, x	are the space coordinates, m;
t	is the time coordinate, sec;
r_0	is the channel radius, m;
D	is the turbulent diffusion coefficient, m^2/sec;
u_0	is the mean flow velocity across channel section, m / sec;
$f(t)$	is the function of concentration in the initial section, %;
$\beta = A/\varphi$;	
A	is the gas exchange per unit surface area, $m^3/sec/m^2$;
φ	is the diffusion coefficient between flow and channel wall, m^2/sec.

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